MATH 3060 Assignment 1 solution

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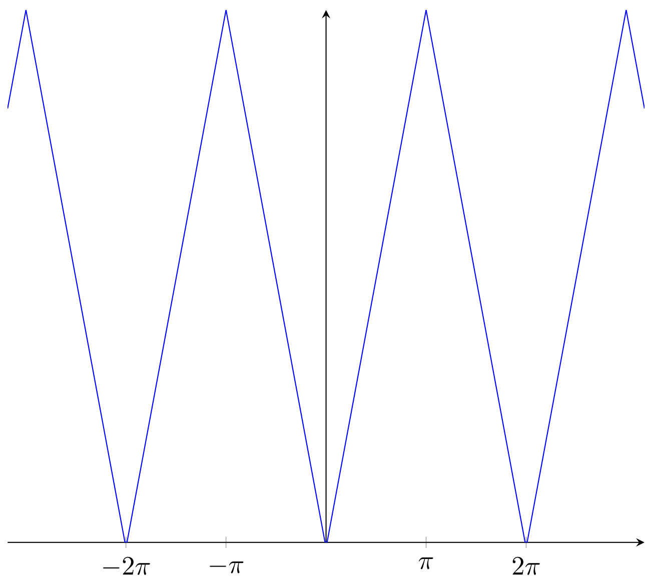
1. (a) Graph:



$b_n = 0$ because f_1 is an even function. On the other hand,

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f_1(x)dx \\ &= \frac{1}{\pi}\int_0^{\pi} xdx \\ &= \frac{\pi}{2} \end{aligned}$$

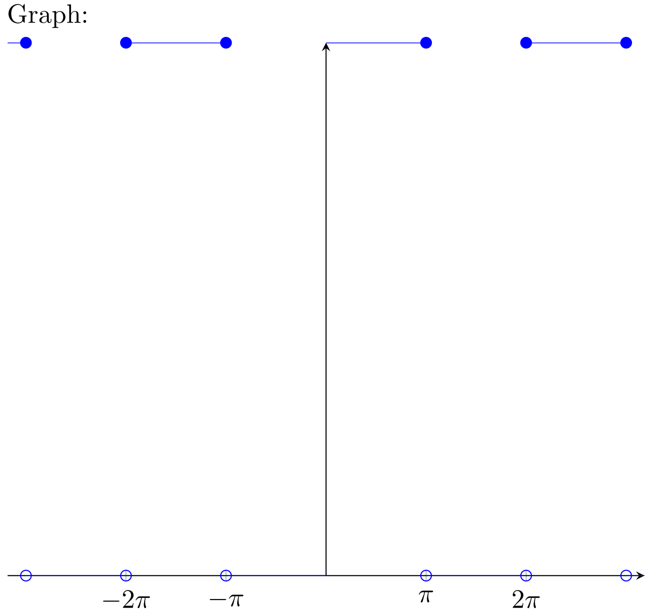
$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f_1(x)\cos nxdx \\ &= \frac{2}{\pi}\int_0^{\pi} \cos nxdx \\ &= \frac{2}{\pi}\left[\frac{1}{n}x\sin x + \frac{2}{n^2}\cos x\right]_0^{\pi} \\ &= \frac{2}{n^2\pi}((-1)^n - 1) \end{aligned}$$

Therefore

$$f_1(x) \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)x.$$

The Fourier series converges to the f_1.

(b) Graph:



$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_0^\pi dx \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^\pi \cos nx dx \\ &= 0 \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_0^\pi \sin nx \\ &= \frac{1-(-1)^n}{n\pi}. \end{aligned}$$

Therefore

$$f_2(x) \sim \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)} \sin(2n-1)x.$$

The limit of the Fourier series is the 2π periodic function

$$\begin{cases} 0, & x \in (-\pi, 0) \\ 1, & x \in (0, \pi) \\ \frac{1}{2}, & x \in \{0, \pi\} \end{cases}$$

(c)

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-2x}e^{-nix}dx \\ &= \frac{1}{2\pi}\left[\frac{-1}{2+ni}e^{-(2+ni)x}\right]_{-\pi}^{\pi} \\ &= (-1)^n\frac{e^{2\pi}-e^{-2\pi}}{2\pi(2+ni)} \\ &= \frac{(-1)^n \sinh 2\pi}{(2+ni)\pi} \end{aligned}$$

Therefore

$$\begin{aligned} f_2(x) &\sim \frac{\sinh 2\pi}{\pi}\sum_{-\infty}^{\infty}\frac{(-1)^n}{2+ni}e^{nix} \\ &= \frac{\sinh 2\pi}{2\pi} + \frac{4\sinh 2\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+4}(2\cos nx - \sin nx) \end{aligned}$$

The limit of the Fourier series is the 2π periodic function

$$\begin{cases} e^{-2x}, & x \in (-\pi, \pi) \\ \cosh 2\pi \end{cases}$$

2. We need to find sequences $\{x_n\}, \{y_n\}$ in $[-\pi, \pi]$ such that $|x_n - y_n| \to 0$ but

$$\frac{|f(x_n) - f(y_n)|}{|x_n - y_n|} \to \infty.$$

We can take

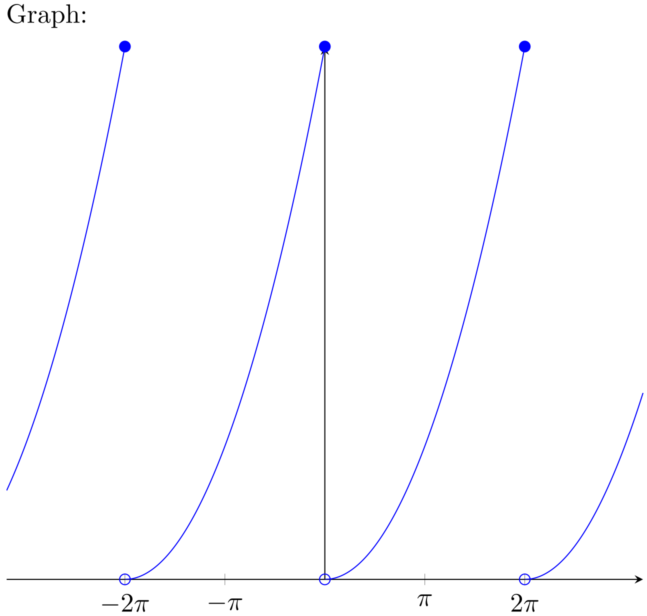
$$x_n = \frac{1}{2n\pi + \pi/2}$$
$$y_n = -\frac{1}{2n\pi + \pi/2},$$

then

$$\frac{f(x_n) - f(y_n)}{x_n - y_n} = \frac{2\sqrt{x_n}}{2x_n} = \frac{1}{\sqrt{x_n}},$$

which $\to \infty$ as $n \to \infty$.

3. Graph:



$$
\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \\
&= \frac{1}{2\pi}\int_{0}^{2\pi} f(x)dx \\
&= \frac{4\pi^2}{3}
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx \\
&= \frac{1}{\pi}\left(\int_{-\pi}^{0} f(x)\cos nx dx + \int_{0}^{\pi} f(x)\cos nx dx\right) \\
&= \frac{1}{\pi}\left(\int_{\pi}^{2\pi} f(x)\cos nx dx + \int_{0}^{\pi} f(x)\cos nx dx\right) \\
&= \frac{1}{\pi}\int_{0}^{2\pi} f(x)\cos nx dx \\
&= \frac{4}{n^2}
\end{aligned}
$$

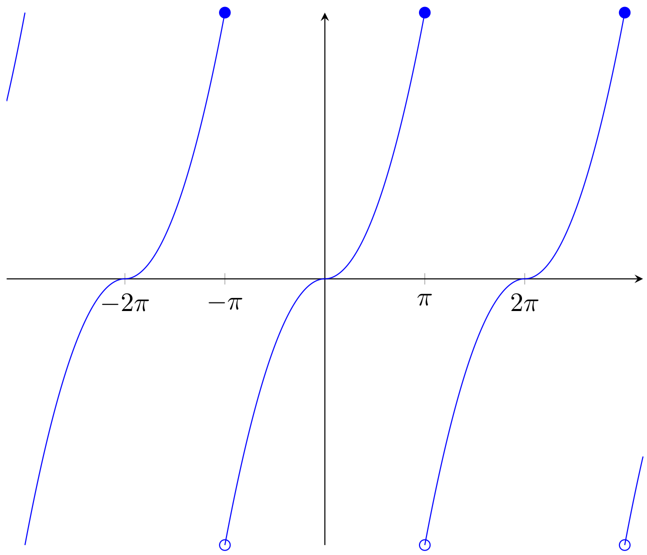
$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx \\
&= \frac{1}{\pi}\int_{0}^{2\pi} f(x)\sin nx dx \\
&= -\frac{4\pi}{n}
\end{aligned}
$$

Therefore,

$$
f(x) \sim \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{\cos nx}{n^2} - 4\pi\sum_{n=1}^{\infty}\frac{\sin nx}{n}
$$

The limit at 0 is $2\pi^2$

4. Graph:



$a_n = 0$ for all $n \geq 0$ because f_1 is odd. On the other hand,

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^\pi x^2 \sin nx dx \\ &= \frac{4((-1)^n - 1)}{n^3\pi} - \frac{2(-1)^n\pi}{n} \end{aligned}$$

Therefore,

$$f_1(x) \sim 2\sum_{n=1}^{\infty}\left(\frac{1}{2n-1} - \frac{4}{(2n-1)^3\pi}\right)\sin(2n-1)x - \pi\sum_{n=1}^{\infty}\frac{1}{n}\sin 2nx$$

The limit at 0 is 0